

Supersymmetric Kähler oscillator in a constant magnetic field.

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Abstract

We propose the notion of the oscillator on Kähler space and consider its supersymmetrization in the presence of a constant magnetic field.

Supersymmetric mechanics attracts permanent interest since its introduction [1]. However, studies focussed mainly on the mechanics with standard $\mathcal{N} = 2$ supersymmetries (see for the review [2] and refs therein). The systems with $\mathcal{N} = 4$ supersymmetries also received much attention: the most general $\mathcal{N} = 4, D = 1, 3$ supersymmetric mechanics described by real superfield actions were studied in Refs. [3, 4] respectively, and those in arbitrary D in Ref.[5]; in [6] $\mathcal{N} = 4, D = 2$ supersymmetric mechanics described by chiral superfield actions were considered. Let us mention also some recent papers on this subject [7]. The study of $\mathcal{N} = 8$ supersymmetric mechanics has been performed recently in Ref.[8].

On the other hand, not enough attention has been paid to systems with non-standard supersymmetry algebra, although they arise in many realistic situations. Some of the systems of that sort were extensively studied by M. Plyushchay [9]. A. Smilga studied the dynamical aspects of “weak supersymmetry” [10] on the simple example of the supersymmetric oscillator. He suggested in this case a nontrivial model of “weak supersymmetric” mechanics, related with quasi-exactly solvable models and the systems with nonlinear supersymmetry.

In the present work we consider the supersymmetrization of a specific model of Hamiltonian mechanics on Kähler manifold $(M_0, g_{a\bar{b}}dz^a d\bar{z}^{\bar{b}})$ interacting with constant magnetic field B , viz

$$\mathcal{H} = g^{a\bar{b}}(\pi_a \bar{\pi}_b + \omega^2 \partial_a K \bar{\partial}_b K), \quad \Omega_0 = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a + iBg_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}}, \quad (1)$$

where $K(z, \bar{z})$ is a Kähler potential of configuration space.

Notice, that the Kähler potential is defined up to holomorphic and antiholomorphic terms,

$$K(z, \bar{z}) \rightarrow K(z, \bar{z}) + U(z) + \bar{U}(\bar{z}), \quad (2)$$

while the Hamiltonian under consideration is not invariant under these transformations. For example, in the limit $\omega \rightarrow 0$ it yields the Hamiltonian

$$\mathcal{H} = g^{a\bar{b}}(\pi_a \bar{\pi}_b + \partial_a U(z) \bar{\partial}_b \bar{U}(\bar{z})). \quad (3)$$

This Hamiltonian admits, in the absence of magnetic field, a $\mathcal{N} = 4$ superextension [12], in the spirit of Alvarez-Gaumé-Freedman [11].

The suggested system could be viewed, in many cases, as a generalization of the oscillator on the Kähler manifold. It includes, as special cases, a few interesting exactly-

- The oscillator on $\mathbb{C}^n = \mathbb{R}^{2n}$,

$$\mathcal{H} = \pi\bar{\pi} + \omega^2 z\bar{z}, \quad (4)$$

corresponding to the choice $U = z^a z^a/2$. The constants of motion defining the hidden symmetries of the system, could be represented as follows:

$$I_{ab}^+ = \pi_a \pi_b + \omega^2 \bar{z}^a \bar{z}^b, \quad I^- = \bar{I}^+, \quad I_{a\bar{b}} = \pi_a \bar{\pi}_b + \omega^2 \bar{z}^a z^b. \quad (5)$$

The symmetry algebra of the system is $u(2n)$. Clearly, these constants of motion are functionally-dependent ones.

- The oscillator on complex projective space \mathbb{CP}^n (for $n > 1$) [14],

$$K = r_0^2 \log(1 + z\bar{z}), \Rightarrow \mathcal{H} = g^{\bar{a}b} \bar{\pi}_a \pi_b + \omega^2 r_0^2 z\bar{z}. \quad (6)$$

This system is also specified, in the absence of magnetic field, by the hidden symmetry given by the constants of motion

$$J_{a\bar{b}} = i(z^b \pi_a - \bar{\pi}_b \bar{z}^a), \quad I_{a\bar{b}} = \frac{J_a^+ J_{\bar{b}}^-}{r_0^2} + \omega^2 r_0^2 \bar{z}^a z^b, \quad (7)$$

where $J_a^+ = \pi_a + (\bar{z}\bar{\pi})\bar{z}^a$, $J_a^- = \bar{J}_a^+$ are the translation generators.

These generators form the nonlinear (quadratic) algebra

$$\begin{aligned} \{J_{a\bar{b}}, J_{c\bar{d}}\} &= i\delta_{a\bar{d}} J_{b\bar{c}} - i\delta_{c\bar{b}} J_{a\bar{d}}, \quad \{I_{a\bar{b}}, J_{c\bar{d}}\} = i\delta_{c\bar{b}} I_{a\bar{d}} - i\delta_{a\bar{d}} I_{c\bar{b}} \\ \{I_{a\bar{b}}, I_{c\bar{d}}\} &= i\omega^2 \delta_{c\bar{b}} J_{a\bar{d}} - i\omega^2 \delta_{a\bar{d}} J_{c\bar{b}} + iI_{c\bar{b}}(J_{a\bar{d}} + J_0 \delta_{a\bar{d}})/r_0^2 - iI_{a\bar{d}}(J_{c\bar{b}} + J_0 \delta_{c\bar{b}})/r_0^2. \end{aligned} \quad (8)$$

- The oscillator on \mathbb{CP}^2 could also be extended to the class of Kähler conifolds, defined by the Kähler potential [15]

$$K = r_0^2 \log(1 \pm (z\bar{z})^\nu), \quad \Rightarrow \quad \mathcal{H} = g^{\bar{a}b} \bar{\pi}_a \pi_b + \omega^2 r_0^2 (z\bar{z})^\nu, \quad (9)$$

where ν is a numerical parameter. Although the corresponding oscillator systems do not have hidden symmetry for $\nu \neq 1$, i.e. on non-constant curvature spaces, they remain exactly-solvable at both the classical [15] and the quantum [16] level. Moreover, for $\nu = 2$ the conic oscillator reduces to the Higgs oscillator on the three-dimensional sphere and pseudosphere interacting with a Dirac monopole field and some specific potential proportional to the squared monopole number.

Notice also, that the Hamiltonian system under consideration, yields, in the “large mass limit”, $\pi_a \rightarrow 0$, the following one,

$$\mathcal{H}_0 = \omega^2 g^{\bar{a}b} \partial_a K \bar{\partial}_b K, \quad \Omega_0 = iB g_{a\bar{b}} dz^a \wedge d\bar{z}^b,$$

which could be easily extended with $\mathcal{N} = 2$ supersymmetry [13].

The supersymmetrization procedure follows closely the steps we performed in [14] for the oscillator on complex projective space. Next we will show that, although the system under consideration does not possess a standard $\mathcal{N} = 4$ superextension, it admits

including, as subalgebras, two copies of $\mathcal{N} = 2$ superalgebras. This nonstandard superextension respects the inclusion of constant magnetic field.

We follow the following strategy. At first, we extend the initial phase space to the a $(2N.2N)_{\mathbb{C}}$ -dimensional superspace equipped with the symplectic structure

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_{\bar{a}} \wedge d\bar{z}^{\bar{a}} + i(Bg_{a\bar{b}} + iR_{a\bar{b}cd}\eta_{\alpha}^c \bar{\eta}_{\alpha}^d) dz^a \wedge d\bar{z}^{\bar{b}} + g_{a\bar{b}} D\eta_{\alpha}^a \wedge D\bar{\eta}_{\alpha}^b \quad . \quad (10)$$

Here $D\eta_{\alpha}^a = d\eta_{\alpha}^a + \Gamma_{bc}^a \eta_{\alpha}^b dz^c$, $\alpha = 1, 2$, and Γ_{bc}^a , $R_{a\bar{b}cd}$ are, respectively, the connection and curvature of the Kähler structure. The corresponding Poisson brackets are defined by the following non-zero relations (and their complex-conjugates):

$$\begin{aligned} \{\pi_a, z^b\} &= \delta_a^b, & \{\pi_a, \eta_{\alpha}^b\} &= -\Gamma_{ac}^b \eta_{\alpha}^c, \\ \{\pi_a, \bar{\pi}_{\bar{b}}\} &= i(Bg_{a\bar{b}} + iR_{a\bar{b}cd}\eta_{\alpha}^c \bar{\eta}_{\alpha}^d), & \{\eta_{\alpha}^a, \bar{\eta}_{\beta}^b\} &= g^{a\bar{b}} \delta_{\alpha\beta}. \end{aligned} \quad (11)$$

The symplectic structure (10) becomes canonical in the coordinates (p_a, χ^k)

$$\begin{aligned} p_a &= \pi_a - \frac{i}{2} \partial_a \mathbf{g}, & \chi_i^m &= e_b^m \eta_i^b : \\ \Omega_{Scan} &= dp_a \wedge dz^a + d\bar{p}_{\bar{a}} \wedge d\bar{z}^{\bar{a}} + iBg_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}} + d\chi_{\alpha}^m \wedge d\bar{\chi}_{\alpha}^{\bar{m}}, \end{aligned} \quad (12)$$

where e_a^m are the einbeins of the Kähler structure: $e_a^m \delta_{m\bar{m}} \bar{e}_b^{\bar{m}} = g_{a\bar{b}}$.

So, in order to quantize the system, one chooses

$$\hat{p}_a = -i \left(\frac{\partial}{\partial z^a} - iB \frac{\partial K}{\partial z^a} \right), \quad \hat{\bar{p}}_{\bar{a}} = -i \left(\frac{\partial}{\partial \bar{z}^{\bar{a}}} + iB \frac{\partial K}{\partial \bar{z}^{\bar{a}}} \right), \quad [\hat{\chi}_{\alpha}^m, \hat{\bar{\chi}}_{\beta}^{\bar{n}}]_{+} = \delta^{m\bar{n}} \delta_{\alpha\beta}.$$

Then, in order to construct the system with the exact $\mathcal{N} = 2$ supersymmetry

$$\{Q_{+}, Q^{-}\} = \mathcal{H}, \{Q_{\pm}, Q_{\pm}\} = \{Q_{\pm}, \mathcal{H}\} = 0, \quad (13)$$

we shall search for the odd functions Q^{\pm} , which obey the equations $\{Q^{\pm}, Q^{\pm}\} = 0$ (we restrict ourselves to the supersymmetric mechanics whose supercharges are *linear* in the Grassmann variables $\eta_i^a, \bar{\eta}_i^{\bar{a}}$). In that case, the Poisson bracket $\{Q_{+}, Q_{-}\}$ yields the $\mathcal{N} = 2$ supersymmetric Hamiltonian.

Let us search for the realization of supercharges among the functions

$$Q^{\pm} = \cos \lambda \Theta_1^{\pm} + \sin \lambda \Theta_2^{\pm}, \quad (14)$$

where

$$\Theta_1^{+} = \pi_a \eta_1^a + i\bar{\partial}_{\bar{a}} W \bar{\eta}_2^{\bar{a}}, \quad \Theta_2^{+} = \bar{\pi}_{\bar{a}} \bar{\eta}_2^{\bar{a}} + i\partial_a W \eta_1^a, \quad \Theta_{1,2}^{-} = \bar{\Theta}_{1,2}^{+}, \quad (15)$$

and λ is some parameter.

Calculating the Poisson brackets of the functions, we get

$$\{Q^{\pm}, Q^{\pm}\} = i(B \sin 2\lambda + 2\omega \cos 2\lambda) \mathcal{F}_{\pm}, \quad (16)$$

$$\{Q^{+}, Q^{-}\} = \mathcal{H}_{SUSY}^0 + (B \cos 2\lambda - 2\omega \sin 2\lambda) \mathcal{F}_3/2. \quad (17)$$

Here and in the following, we use the notation

$$\mathcal{H}^0 = \mathcal{H} - B - \omega \bar{\pi}_{\bar{a}}^a \pi_{\alpha}^c \bar{\eta}_{\alpha}^d - iW - \omega \pi_{\alpha}^a \pi_{\alpha}^b + iW - \bar{\omega} \bar{\pi}_{\bar{a}}^a \bar{\pi}_{\bar{a}}^b + B + i g_{a\bar{b}} \eta_{\alpha}^a \bar{\eta}_{\alpha}^b \quad (18)$$

where \mathcal{H} denotes the oscillator Hamiltonian (see the expression in (1)) and

$$\mathcal{F}_i = \frac{i}{2} g_{ab} \eta_\alpha^a \bar{\eta}_\beta^b \sigma_{(i)\alpha\bar{\beta}}, \quad \mathcal{F}_\pm = \mathcal{F}_1 \pm \mathcal{F}_2. \quad (19)$$

One has, then

$$\{Q^\pm, Q^\pm\} = 0 \Leftrightarrow B \sin 2\lambda + 2\omega \cos 2\lambda = 0, \quad (20)$$

so that

$$\lambda = \lambda_0 + (\alpha - 1)\pi/2, \quad \alpha = 1, 2. \quad (21)$$

Here the parameter λ_0 is defined by the expressions

$$\cos 2\lambda_0 = \frac{B/2}{\sqrt{\omega^2 + (B/2)^2}}, \quad \sin 2\lambda_0 = -\frac{\omega}{\sqrt{\omega^2 + (B/2)^2}}. \quad (22)$$

Hence, we get the following supercharges:

$$Q_\alpha^\pm = \cos \lambda_0 \Theta_1^\pm + (-1)^\alpha \sin \lambda_0 \Theta_2^\pm, \quad (23)$$

and the pair of $\mathcal{N} = 2$ supersymmetric Hamiltonians

$$\mathcal{H}_{SUSY}^\alpha = \{Q_\alpha^+, Q_\alpha^-\} = \mathcal{H}_{SUSY}^0 - (-1)^\alpha \sqrt{\omega^2 + (B/2)^2} \mathcal{F}_3 \quad (24)$$

Notice that the supersymmetry invariance is preserved in the presence of the constant magnetic field.

Calculating the commutators of Q_1^\pm and Q_2^\pm we get

$$\{Q_1^\pm, Q_2^\pm\} = 2\sqrt{\omega^2 + (B/2)^2} \mathcal{F}_\pm, \quad \{Q_1^+, Q_2^-\} = 0, \quad (25)$$

where the Poisson brackets between \mathcal{F}_\pm , and Q_α^\pm look as follows:

$$\{Q_\alpha^\pm, \mathcal{F}_\pm\} = 0, \quad \{Q_\alpha^\pm, \mathcal{F}_\mp\} = \pm \epsilon_{\alpha\beta} Q_\beta^\pm, \quad \{Q_\alpha^\pm, \mathcal{F}_3\} = \pm i Q_\alpha^\pm. \quad (26)$$

The whole superalgebra reads

$$\begin{aligned} \{Q_\alpha^\pm, Q_\beta^\pm\} &= 2\Lambda \epsilon_{\alpha\beta} \mathcal{F}_\pm, & \{Q_\alpha^\pm, Q_\beta^\mp\} &= \delta_{\alpha\beta} \mathcal{H}_{SUSY}^0 - \Lambda \sigma_{\alpha\beta}^3 \mathcal{F}_3, \\ \{Q_\alpha^\pm, \mathcal{F}_\pm\} &= 0, & \{Q_\alpha^\pm, \mathcal{F}_\mp\} &= \pm \epsilon_{\alpha\beta} Q_\beta^\pm, & \{Q_\alpha^\pm, \mathcal{F}_3\} &= \pm i Q_\alpha^\pm, \\ \{\mathcal{F}_\pm, \mathcal{F}_\mp\} &= i \mathcal{F}_3, & \{\mathcal{F}_\pm, \mathcal{F}_3\} &= \pm i \mathcal{F}_\pm. \end{aligned} \quad (27)$$

where

$$\Lambda = \sqrt{\omega^2 + (B/2)^2}. \quad (28)$$

Let us notice the ω and B appear in this superalgebra in a symmetric way, via the factor $\sqrt{\omega^2 + (B/2)^2}$.

This superalgebra could be represented in a bit more convenient form, if introduce

$$S_1^\pm \equiv Q_1^\pm, \quad S_2^\pm = Q_2^\mp. \quad (29)$$

In this notation, it reads

All other commutators vanish.

This is precisely the weak supersymmetry algebra considered by A. Smilga [10]. In the particular case $\omega = 0$ it yields the $\mathcal{N} = 4$ supersymmetric mechanics broken by a constant magnetic field.

Remark 1. In the case of the oscillator on \mathbb{C}^n we can smoothly relate the above supersymmetric oscillator with a $N = 4$ oscillator, if choose

$$K = \cos \gamma \, z \bar{z} + \sin \gamma \, (z^2 + \bar{z}^2)/2, \quad \gamma \in [0, \pi/2]. \quad (31)$$

Hence,

$$\mathcal{H} = \pi \bar{\pi} + \omega_0^2 z \bar{z} + \sin 2\gamma \, \omega_0^2 (z^2 + \bar{z}^2)/2, \quad (32)$$

i.e. for $\gamma = 0, \pi/2$ we have a standard harmonic oscillator, while for $\gamma \neq 0, \pi/2$ we get the anisotropic one, which is equivalent to two sets of n one-dimensional oscillators with frequencies $\omega_0 \sqrt{1 \pm \sin 2\gamma}$. The frequency ω appearing in the superalgebra, is of the form: $\omega = \omega_0 \cos \gamma$.

Remark 2. In a similar way, we can consider the supersymmetrization of the two-dimensional noncommutative oscillator in the constant magnetic field [17]. For this purpose, we define the Poisson brackets [18]

$$\{\pi, z\} = 1, \quad \{z, \bar{z}\} = i\theta, \quad \{\pi, \bar{\pi}\} = iB, \quad \{\eta_\alpha, \bar{\eta}_\beta\} = \delta_{\alpha\beta} \quad (33)$$

and choose the following supercharges:

$$S_\alpha^+ = \pi \eta_\alpha + i \epsilon_{\alpha\beta} \bar{z} \bar{\eta}_\beta, \quad S_\alpha^- = \bar{S}_\alpha^+. \quad (34)$$

Calculating the Poisson brackets, we get the same superalgebra as above, with the following Hamiltonian and Λ parameter:

$$\mathcal{H}_{SUSY}^0 = \pi \bar{\pi} + \omega^2 z \bar{z} - i\omega \eta_1 \eta_2 + i\omega \bar{\eta}_1 \bar{\eta}_2 + \frac{i}{2}(B + \theta\omega^2)\eta_\alpha \bar{\eta}_\alpha, \quad \Lambda = B - \theta\omega^2. \quad (35)$$

Hence, the exact $\mathcal{N} = 4$ supersymmetry is realized only under the choice of the parameter

$$\Lambda = 0 \quad \Leftrightarrow \quad B = \theta\omega^2. \quad (36)$$

This is not surprising, since under this choice of Λ the underlying system is equivalent to the isotropic oscillator [17].

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